Climate risk and model uncertainty

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Main contents

What's risk?

The term “risk” is used differently in everyday life and in literature, depending on the context:

- In colloquial language: occurrence of “unfavourable” events with adverse (economic) consequences.
- Concise Oxford English Dictionary: “hazard, a chance of bad consequences, loss or exposure to mischance”.
- The standard “ISO 31000 - Risk Management” describes risk as the “effect of uncertainty on objectives”.
- Keywords: decisions, uncertainty, events, consequences.

Climate risk

- The Earth's climate is changing: average temperatures rise, acute phenomena such as heat waves and floods grow in frequency and severity, and chronic phenomena, such as drought and rising sea levels, intensify.
- First fundamental question: which actions should be tackled in order to mitigate climate change?
- Second fundamental question: how can climate change impact socioeconomic and financial systems across the world in the next decades?
- Climate change risk assessment involves formal analysis of the consequences, likelihoods and responses to the impacts of climate change and the options for addressing them.
- In this lecture we will focus more on the impact of climate risk in financial systems.

Climate risk in finance

Financial institutions face today face a two-sided climate risk: a physical impact risk and a policy risk .

- Many possible catastrophic events are linked to climate change: fires (California 2018, Australia 2020), hurricanes, floods, and probably also pandemics like Covid-19. These events may cause dramatic losses in different ways.
- Across the world, we see a tightening of climate policies and regulations to shift the economy away from fossil fuels. The restructuring is accelerated by the Paris Agreement, which sets clear aspirations to limit global warming to 1.5 or 2 degrees Celsius, and will affect all sectors and future investment patterns for global financial capital.

Both physical and policy risks can result in real financial impacts to companies and assets.

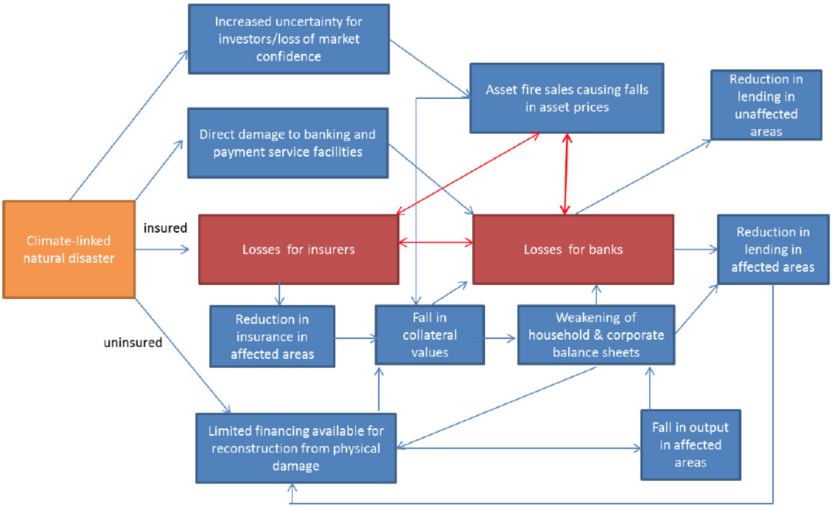
Effects of a natural disaster on the financial system

- Natural disasters may destroy the physical capital, forcing the companies directly affected to allocate financial resources to reconstruction. Such a diversion of resources has the effect of increasing debt, thus reducing the resources available for consumption and investment.
- Environmental shocks may increase the number of non-performing loans in the portfolio of banks that are particularly exposed to households or businesses in the areas most at risk. This could induce banks to restrict the supply of credit, which would potentially affect the effectiveness of the credit channel of monetary policy.
- If the damaged infrastructures are not insured, the effects of natural events take away more resources from the people involved and may lead to a more significant reduction in the value of the collateral pledged for credit.
- In turn, a reduction in the value of collateral, associated with an increase in the financial vulnerability of the companies hit by the shock, could increase both the possibility of default and the amount of the loss that the bank must bear in case of a borrower's default.

Effects on insurance

- If the companies affected by natural disasters are insured, this can have a big repercussion on the institutions whose business is taking on these kinds of risks, i.e. insurance companies.
- A deterioration in the financial position of insurance companies could in turn affect financial stability if they stop providing certain services or the value of their securities abruptly decreases, thus negatively affecting the situation of other financial institutions that hold them in their portfolio.
- When insurance have to bear huge losses due to catastrophic events, re-insurance companies might also be distressed.

Effects of a natural disaster on the financial system



Source: Batten et al. (2016).

Transition risk

- A second risk comes from the commitments made by the international community in order to decrease the atmospheric concentration of greenhouse gases at a level that allows the increase in temperature to be kept below 2°C compared with pre-industrial levels.
- A sudden drop in the value of reserves and related infrastructures could start a race to sell the securities of energy companies, with consequences that could permanently affect the path to global economic growth.
- Moreover, the transition could be inflationary, because climate policies may require the use of alternative energy sources that are currently more expensive, or the introduction of carbon pricing systems that affect prices and economic activities (e.g. the imposition of a carbon tax)

So: we have to measure a given risk..

- Kloman 1990: “risk management is a discipline for living with the possibility that future events may cause adverse effects”
- Quantitative approaches to risk assessment often identify risk with the fluctuation of a value variable.
- Two kinds or approaches:
  1. One-sided approaches: only consideration of “unfavourable” deviations
  2. Two-sided approaches: consideration of both “favourable” and ‘unfavourable” deviations
- Examples of risk measurement related to climate risk in finance:
  - an insurance company might want to assess the risk of big losses in most exposed areas (i.e., Florida with hurricanes);
  - a bank might want to quantify its exposure to transition risk.

A first approach to risk measurement in Financial mathematics

- Let $(\Omega, \mathcal{F}, P)$ be a probability space.
- $\mathcal{X} = \{X$ random variable on $(\Omega, \mathcal{F}, P)$, with some integrability condition w.r.t. $P\}$
- X stands for the value of a financial position at the end of a given period (for example, liquidation time of positions).
- A risk measure ρ is a functional

$$\rho : \mathcal{X} \to \mathbb{R},$$

  assigning a risk $\rho(X)$ to the financial position represented by X.
- In financial applications, a rational decision maker tries to find a position $X \in \mathcal{X}$, with possibly some constraints, that minimizes $\rho(X)$.

- Risk measures are defined either in relation to the financial position X or to the loss $L = -X$.
- This difference must be taken into account in practical work and when applying results from the literature.
- In this lecture, the risk for us will be usually given in terms of financial positions.

Some examples of risk measures

In the examples below, $\mathbb{E}[\cdot]$ denotes expectation with respect to P, i.e. $\mathbb{E}[X] = \int_\Omega X dP$.

- Variance:

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

- Normalized standard deviation:

$$\tilde{\sigma}(X) = \frac{\sqrt{Var(X)}}{\mathbb{E}[X]}.$$

  Intuition: random variables with a large expected value often have a large variance or standard deviation

- Semivariance:

$$Var_+(X) = \mathbb{E}\left[\left((\mathbb{E}[X] - X)^+\right)^2\right].$$

  Note: only shortfalls $X < \mathbb{E}[X]$ are taken into account.

- Value at Risk at level $\alpha \in (0,1)$ of a financial position X:

$$VaR_\alpha(X) := \inf\{m \in \mathbb{R} : P(X + m < 0) \leq \alpha\}.$$

  Interpretation: smallest amount of money (“risk capital”) that must be added to X so that the probability of bankruptcy is $\leq \alpha$.

But: are these risk measures appropriate for climate risk?

- Problem: the climate change case illustrates particularly well a situation in which the probabilistic model, i.e., the probability measure P, is neither explicitly given nor can it be adequately approximated or inferred with the available data and current scientific methods: deep model uncertainty.
- These uncertainties arise from both the extreme complexity of the climatic system and our inability to perfectly capture the way our socioeconomic system would respond and adapt to climate change.
- This is particularly the case when we consider situations with potential catastrophic consequences, such as the collapse of the Atlantic thermohaline circulation, the melting of the Antarctic ice sheet or the loss of the Amazon rainforest. Such catastrophic events (also called tipping points) have not been encountered in recent history, and therefore their likelihood of occurrence is extremely difficult to assess.

How to deal with this issue?

- In view of this disagreement among experts or models, how should a rational policy decision maker proceed?
- If one follows the traditional Bayesian/subjective risk minimazation approach, one will simply aggregate the models by averaging them into a single representative model.
- The problem with this approach is that the decision maker considers the resulting aggregated model in exactly the same way as one would consider an equivalent objective model representing a specific risk, and model uncertainty has therefore no impact on the decision-making process.
- Ellsberg (1961) showed through different experiments that the choices of individuals cannot be rationalized under the traditional Bayesian expected utility paradigm, and that individuals usually manifest aversion toward situations in which probabilities are not perfectly known.

Let's start with an example: Ellsberg paradox

Urn with 90 balls: 30 red, 60 black *OR* white.
People have been required to answer the following questions:

1. do you prefer to receive $100\$$ when you:
   a. draw a red ball
   b. draw a white ball
2. do you prefer to receive $100\$$ when you:
   a. draw a red or black ball
   b. draw a white or black ball

Try to guess the most common answers..

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Try to guess the most common answers..

- (a) to point 1, (b) to point 2.
- But why? Relying on utility theory, if you prefer red to white you also prefer [red or black] to [white or black]!
- Possible reason: people are averse to model uncertainty.
- Let's go more into details..

Risk vs uncertainty (Knight, 1921)

- There's a difference between two types of "imperfect knowledge":
  1. risk (or measurable uncertainty) $\rightarrow$ situations in which the distribution of the target random variables is known;
  2. (Knightian, model, or not measurable) uncertainty $\rightarrow$ the distribution of the target random variables is not known. This is the case for many issues related to climate risk.
- Think about the previous example: if you win when you draw a red ball, your gamble is based on a distribution you know: $P(\text{win}) = \frac{1}{3}$. This is not the case if you win when the white ball is drawn. Same thing for the second choice.
- The example shows that people do no treat these kinds of uncertainty in the same way: ambiguity aversion.

Model uncertainty and Financial mathematics

- Standard procedure: modelling under the usual concept of “Risk”:
  - Tacit assumption: a fixed probability measure P, and thus the distribution of the underlying random variables/sources of risk, is known.
  - Example in financial mathematics: we specify the dynamics of some stochastic processes with respect to a fixed probability P and we price derivatives based on those dynamics.
- The assumption above is not realistic for climate risk (as well as in other fields of finance).
- Approach under model uncertainty: probabilities are unknown for financial market events $\rightarrow$ Increased awareness of the problems that can result from excessive reliance on a specific probabilistic model is needed.

Approach under model uncertainty

- Instead of a reference measure P, consider a family $\mathcal{P}$ of possible probability measures. Each element of $\mathcal{P}$ reflects a possible different model, which gives rise to a different probability distribution.
- Extension, and robustification, of the classical portfolio theory.
- Example: utility maximization under model uncertainty
  - S stochastic process with log-normal returns R_k, i.e.,

$$S_T = S_0 e^{R_1 + R_2 + \cdots + R_T}.$$

  - Introduce a family of probability measures to express uncertainty about returns:

$$\mathcal{P} := \{P^\mu | \mu \in [a, b] \text{ and } R_1, R_2, \ldots, R_T \text{ i.i.d. }, R_k \sim \mathcal{N}(\mu, \sigma^2) \text{ under } P^\mu\}.$$

  - The maximization of the expected utility of a financial position X involving S and a risk-free asset can be achieved by

$$\text{maximize } \inf_{P \in \tilde{P}} \mathbb{E}^{\tilde{P}}[u(X)], \quad X \in \mathcal{X},$$

  $u(\cdot)$ utility function, $\mathcal{X}$ family of financial positions: maxmin approach.

Model uncertainty and risk management

- There are different possible ways to deal with model uncertainty in risk management (and so in particular with climate risk).
- A key idea is that risk measures should be robust with respect to model uncertainty.
- There is not a unique notion of robustness for a risk measure. In this lecture we will see two of them:
  - robust representation: a risk measure has a robust representation if it can be characterized without referring to a given a priori measure.
  - robustness in the sense of Embrechts, Schied and Wang (2019): a risk measure ρ is robust if the minimization of $\rho(X)$ does not strongly depend on small changes in the distribution of X.
- We first focus on the first notion above. We start from a characterization of risk measures with some desired properties.
- The next section is based on the paper Robust Preferences and Convex Measures of Risk, Föllmer and Schied, 2002.

Motivation

- At the turn of the millennium, the weaknesses of Value at Risk led to the development of an axiomatic theory of risk measures:
  - P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk, Mathematical Finance 9, 1999;
  - H. Föllmer, A. Schied, Convex measures of risk and trading constraints. Finance $\&$ Stochastics 4, 2000.
- Core ideas:
  1. The risk of a position X has to be quantified as the minimum capital which must be added to X so that the position becomes acceptable (e.g. from the point of view of a supervisory authority)
  2. Diversification must be incentivated: subadditivity/convexity

The setting

- Take a measurable space $(\Omega, \mathcal{F})$, standing for possible scenarios. Note: no probability measure is specified!
- A financial position is modelled by a random variable $X : \Omega \to \mathbb{R}$: $X(\omega)$ is the discounted value of the position at the end of a given period (liquidation time, as before) in the scenario ω.
- The space $\mathcal{X}$ of all possible positions is a linear subspace of measurable functions on $(\Omega, \mathcal{F})$, which contains the constants.

Monetary risk measures

A functional $\rho : \mathcal{X} \to \mathbb{R}$ is called monetary risk measure if it satisfies the following properties:

1. no position is "infinitely good": $\rho(X) > -\infty$ for all $X \in \mathcal{X}$;
2. for every constant $m \in \mathbb{R}$ it holds $\rho(m) < +\infty$;
3. monotonicity: if $X \leq Y$ (i.e., $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$), it holds $\rho(X) \geq \rho(Y)$;
4. cash invariance: for every $m \in \mathbb{R}$ it holds $\rho(X + m) = \rho(X) - m$: if a capital m is added to a position X, the risk of new position $X + m$ is reduced by amount m.

A set $\mathcal{A} \subset \mathcal{X}$ is said to be an acceptance set if:

1. $\mathcal{A} \cap \{\text{constant functions}\} \neq \emptyset$: $\exists m \in \mathbb{R}$ such that having m is acceptable;
2. For all $X \in \mathcal{X}$ there exists $m \in \mathbb{R}$ such that $X + m \neq \mathcal{A}$: no position is "infinitely good";
3. $\mathcal{A}$ is monotone in the sense that $X \in \mathcal{A}$, $Y \in \mathcal{X}$ and $Y \geq X$ implies $Y \in \mathcal{A}$.

Proposition

Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set. Thus the functional $\rho_{\mathcal{A}} : \mathcal{X} \to \mathbb{R}$ defined by

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$$

is a monetary risk measure.

Proposition

Let a functional $\rho : \mathcal{X} \to \mathbb{R}$ be a monetary risk measure. Thus the set $\mathcal{A}_\rho$ defined by

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}$$

is an acceptance set.

Convex risk measures

Remember: diversification should not increase risk!

Definition

A monetary risk measure ρ is called a convex risk measure if for every $\lambda \in [0,1]$, $X, Y \in \mathcal{X}$ it holds

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

Proposition

A monetary risk measure is convex if and only if for every $\lambda \in [0,1]$, $X, Y \in \mathcal{X}$ it holds

$$\rho(\lambda X + (1-\lambda)Y) \leq \max(\rho(X), \rho(Y)).$$

Proposition

A monetary risk measure ρ is convex if and only if $\mathcal{A}_\rho$ is a convex set.

Coherent risk measures and subadditivity

Definition

A convex risk measure ρ is called coherent risk measure if it is positive homogenous, i.e., if for every $\lambda \geq 0$, $X \in \mathcal{X}$ it holds

$$\rho(\lambda X) = \lambda \rho(X).$$

Proposition

A coherent risk measure is subadditive, i.e., for every $X, Y \in \mathcal{X}$ it holds

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

Proposition

A monetary risk measure ρ is coherent if and only if $\mathcal{A}_\rho$ is a convex cone.

What about model uncertainty?

- Note that for now we have not fixed any probability measure, so no model for our risky financial position X.
- On the other hand, no notions of robustness with respect to model uncertainty have been specified.
- This is what we want to do now.

Definition

A risk measure ρ admits a robust representation if for every $X \in \mathcal{X}$ it holds

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\},$$

where

$$\mathcal{M} = \{\text{probability measures } Q \text{ on } (\Omega, \mathcal{F}) \text{ such that } \mathbb{E}^Q[X] \text{ is finite for every } X \in \mathcal{X}\}.$$

The functional $\alpha : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ is called penalty function.

Interpretation

- The elements of $\mathcal{M}$ can be interpreted as possible probabilistic models, which are taken more or less “seriously” according to the size of the penalty $\alpha(Q)$.
- The value $\rho(X)$ is computed as the worst case expectation taken over all models $Q \in \mathcal{M}$ and penalized by $\alpha(Q)$.

Robust representation for coherent measures

Proposition

A risk measure ρ satisfying the representation above is convex.

Proof

Let $\lambda \in (0,1)$, and suppose that ρ has the representation

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\},$$

with $\alpha : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$. Then for every $X, Y \in \mathcal{X}$ and $\lambda \in (0,1)$ it holds

$$\begin{aligned}
\rho(\lambda X + (1-\lambda)Y) &= \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-\lambda X - (1-\lambda)Y] - \alpha(Q) \right\} \\
&= \sup_{Q \in \mathcal{M}} \left\{ \lambda \mathbb{E}^Q[-X] + (1-\lambda)\mathbb{E}^Q[-Y] - \lambda\alpha(Q) - (1-\lambda)\alpha(Q) \right\} \\
&= \sup_{Q \in \mathcal{M}} \left\{ \lambda \left(\mathbb{E}^Q[-X] - \alpha(Q) \right) + (1-\lambda) \left(\mathbb{E}^Q[-Y] - \alpha(Q) \right) \right\} \\
&\leq \lambda \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\} + (1-\lambda) \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-Y] - \alpha(Q) \right\} \\
&= \lambda\rho(X) + (1-\lambda)\rho(Y).
\end{aligned}$$

Proposition

- A risk measure ρ which admits a robust representation is coherent if and only if the penalty function α only takes the values 0 and ∞, i.e.

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[-X]$$

where $\mathcal{Q} = \{Q \in \mathcal{M} : \alpha(Q) = 0\}$.

Remark

In the following examples a probability measure P is fixed in $(\Omega, \mathcal{F})$ and the linear space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ is considered. All risk measures are initially defined on $\mathcal{X}$, but have canonical extensions to larger spaces.
The expectation will be always taken with respect to P unless differently specified, i.e.

$$\mathbb{E}[X] = \int_\Omega X dP.$$

Note that since we consider bounded random variables, the set $\mathcal{M}$ introduced above is the space of probability measures in $(\Omega, \mathcal{F})$.

- The Value at Risk at level λ is a monetary risk measure with acceptance set

$$\mathcal{A}_\lambda = \{X \in \mathcal{X} : P(X < 0) \leq \lambda\}.$$

- In terms of capital requirement:

$$\begin{aligned} VaR_\lambda(X) &= \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_\lambda\} \\ &= \inf\{m \in \mathbb{R} : P(X + m < 0) \leq \lambda\}. \end{aligned}$$

- Note: Value at Risk is a positive homogenous monetary measure, but no convex!
- It follows that not only Value at Risk does not reward diversification, but from the proposition we have seen it also fails to have a robust representation.

Definition

The Average Value at Risk at level $\lambda \in (0,1]$ for a position X is

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_\beta(X)d\beta.$$

- As opposed to Value at Risk, it takes into account extreme losses.
- Since $\lambda \to VaR_\lambda$ is non-decreasing, it holds

$$AVaR_\lambda(X) \geq VaR_\lambda(X):$$

  Average Value at Risk is more conservative with respect to Value at Risk.
- It is a coherent risk measure with robust representation

$$AVaR_\lambda(X) = \sup_{Q \in \mathcal{Q}_\lambda(P)} \mathbb{E}^Q[-X]$$

  with

$$\mathcal{Q}_\lambda(P) := \left\{ Q \in \mathcal{M}, Q \ll P : \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}.$$

Utility-based shortfall risk

Definition

Let $\ell : \mathbb{R} \to \mathbb{R}$ be a convex and increasing function, and take $r_0 > \inf_{x\in\mathbb{R}}\{\ell(x)\}$. The utility-based shortfall risk ρ for a position $X \in \mathcal{X}$ is defined as

$$\rho = \inf\{m : X + m \in \mathcal{A}\}$$

where

$$\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[\ell(-X)] \leq r_0\}.$$

Remark

The acceptance set $\mathcal{A}$ can be written in the form

$$\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[u(X)] \geq 0\}$$

for the utility function $u(x) := r_0 - \ell(-x)$. From this the name “utility-based shortfall risk”.

- The acceptance set $\mathcal{A}$ is convex, so ρ is a convex risk measure.
- If X has a continuous distribution and if ℓ is continuous, $m = \rho(X)$ is the unique solution to the equation

$$\mathbb{E}[\ell(-X - m)] = r_0.$$

  This can be solved by numerical methods.
- ρ admits a robust representation

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \{\mathbb{E}^Q[-X] - \alpha(Q)\}$$

  where $\mathcal{M}_1(P) = \{Q \in \mathcal{M}, Q \ll P\}$ and

$$\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left(r_0 + \mathbb{E}\left[\ell^* \left(\lambda \frac{dQ}{dP} \right) \right] \right),$$

  where $\ell^*(z) := \sup_{x \in \mathbb{R}} \{zx - \ell(x)\}$ is the Fenchel-Legendre transform.

Definition

For a fixed probability measure P and a parameter $\gamma > 0$, the entropy penalty function is defined as $\alpha(Q) := \frac{1}{\gamma} H(Q|P)$, where

$$H(Q|P) := \begin{cases} \mathbb{E}^Q \left[\ln \frac{dQ}{dP} \right] & \text{if } Q \ll P \\ +\infty & \text{otherwise} \end{cases}$$

Interpretation: the more a measure Q "diverges" from P, the more it get penalized.

Definition

For a fixed probability measure P and a parameter $\gamma > 0$, the entropic risk measure for a position X is defined by the robust representation with respect to the entropy penalization function defined above:

$$e_\gamma(X) := \sup_{Q \in \mathcal{M}} \{ \mathbb{E}^Q[-X] - \alpha(Q) \}.$$

More on the entropic risk measure

- It can be seen that

$$H(Q|P) = \sup_{X \in L^\infty(\Omega,\mathcal{F},P)} \{\mathbb{E}^Q[-X] - \ln \mathbb{E}[e^{-X}]\}.$$

- It follows the explicit representation

$$e_\gamma(X) = \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma X}].$$

- Define the loss function $\ell(x) = e^{\gamma x}$ and the utility function $u(x) = 1 - e^{-\gamma x}$. Thus it holds

$$\mathcal{A} = \{X \in \mathcal{X} | e_\gamma(X) \leq 0\} = \{X \in \mathcal{X} | \mathbb{E}[\ell(-X)] \leq 1\} = \{X \in \mathcal{X} | \mathbb{E}[u(X)] \geq 0\}.$$

- Then, the entropic risk measure is a special case of the utility-based shortfall risk measure

The setting

- We have seen some examples of risk measures with a robust representation in the specific setting where a probability measure is fixed.
- We now want to give some more general results.
- From now on we assume that $\mathcal{X}$ is the linear space of all bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$.
- As before, denote by $\mathcal{M}$ the set of all probability measures on $(\Omega, \mathcal{F})$.
- Moreover, we introduce the larger set $\mathcal{M}_F$ of all finitely additive and non-negative set functions Q on $\mathcal{F}$ which are normalized to $Q[\Omega] = 1$. Note that these are not necessarily probability measures, since probability measures must satisfy countable additivity.

Representation with respect to $\mathcal{M}_F$

Theorem

Any convex risk measure ρ on $\mathcal{X}$ admits the representation

$$\rho(X) = \max_{Q \in \mathcal{M}_F} \left(\mathbb{E}^Q[-X] - \alpha_{\min}(Q) \right), \quad X \in \mathcal{X},$$

where the penalty functional $\alpha_{\min}$ is given by

$$\alpha_{\min}(Q) := \sup_{X \in \mathcal{A}_\rho} \mathbb{E}^Q[-X], \quad Q \in \mathcal{M}_F,$$

with

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}.$$

Moreover, $\alpha_{\min}$ is the minimal penalty function which represents ρ, i.e., any penalty function α for which

$$\rho(X) = \max_{Q \in \mathcal{M}_F} \left(\mathbb{E}^Q[-X] - \alpha(Q) \right), \quad X \in \mathcal{X},$$

satisfies $\alpha(Q) \geq \alpha_{\min}(Q)$ for all $Q \in \mathcal{M}_F$.

Representation with respect to $\mathcal{M}_F$ for coherent risk measures

Corollary

The minimal penalty function $\alpha_{\min}$ of a coherent risk measure ρ takes only the values 0 and $+\infty$. In particular,

$$\rho(X) = \max_{Q \in \mathcal{Q}_{\max}} \mathbb{E}^Q[-X], \quad X \in \mathcal{X},$$

for the set

$$\mathcal{Q}_{\max} := \{Q \in \mathcal{M}_F : \alpha_{\min} = 0\}.$$

- We have nice results about the (robust) representation of a convex risk measure with respect to $\mathcal{M}_F$, and basically no more assumptions are needed.
- However, we are interested in the case where ρ admits a representation in terms of (countably additive) probability measures, i.e., it can be represented by a penalty function α which is infinite outside the set $\mathcal{M}$:

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left(\mathbb{E}^Q[-X] - \alpha(Q) \right).$$

In this case, one can no longer expect that the supremum above is attained, see the example in the next slide.

Example

Remember that $\mathcal{M}$ denotes the set of *all* probability measures in $(\Omega, \mathcal{F})$. So it contains all Dirac measures δ_ω for $\omega \in \Omega$, given by

$$\delta_\omega(\omega') = \begin{cases} 1 & \text{if } \omega' = \omega \\ 0 & \text{if } \omega' \neq \omega. \end{cases}$$

It holds $\mathbb{E}^{\delta_\omega}[X] = \int_\Omega X(\omega') d\delta_\omega(\omega') = X(\omega)$.
Thus we have

$$\rho_{\max}(X) := \sup_{Q \in \mathcal{M}} \mathbb{E}^Q[-X] = \sup_{\omega \in \Omega}(-X(\omega)) = -\inf_{\omega \in \Omega} X(\omega) \quad \text{for all } X \in X.$$

Thus, if X does not attain its infimum, there exists no probability measure Q such that $\mathbb{E}^Q[-X] = \rho_{\max}(X)$.

A sufficient condition for having a robust representation with the maximum

Proposition

Let ρ be a convex risk measure which is continuous from below in the sense that

$$\rho(X_n) \searrow \rho(X) \quad \text{whenever} \quad X_n \nearrow X,$$

and suppose that α is any penalty function on $\mathcal{M}_F$ representing ρ, i.e., such that

$$\rho(X) = \max_{Q \in \mathcal{M}_F} \left(\mathbb{E}^Q[-X] - \alpha(Q) \right), \quad X \in \mathcal{X}.$$

Then α is concentrated on probability measures in the usual sense, i.e.,

$$\alpha(Q) < \infty \implies Q \text{ is a probability measure}.$$

A sufficient condition for having a robust representation with the maximum

Remark

The proposition above implies that if ρ is a convex risk measure which is also continuous from below, it can be represented as the *maximum*

$$\rho(X) = \max_{Q \in \mathcal{M}} \left(\mathbb{E}^Q[-X] - \alpha_{\min}(Q) \right).$$

However, the example we have seen shows that not all the convex risk measures with a robust representation

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left(\mathbb{E}^Q[-X] - \alpha_{\min}(Q) \right).$$

are represented by the maximum. So this condition is not necessary for having a robust representation.

Setting for a weaker sufficient condition for a robust representation

- We assume now that Ω is a Polish space, i.e., a separable topological space admitting a complete metric.
- We also suppose $\mathcal{F}$ to be the Borel σ-algebra.
- As before, X is the linear space of all bounded measurable functions on $(\Omega, \mathcal{F})$, and we denote by $C_b(\Omega)$ the subspace of bounded continuous functions on Ω.

Definition

A convex risk measure ρ on $\mathcal{X}$ is called tight if there exists an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets of Ω such that

$$\rho(\lambda \mathbf{1}_{\{K_n\}}) \searrow \rho(\lambda) \quad \text{for all } \lambda \geq 1.$$

A weaker condition for a robust representation

Theorem

Let ρ be a convex risk measure on $\mathcal{X}$. Then the following conditions are equivalent:

- (i) ρ is tight
- (ii) ρ is continuous from below in $C_b(\Omega)$, i.e., if $(X_n)_{n\in\mathbb{N}}$ is a sequence in $C_b(\Omega)$ such that $X_n \nearrow X \in C_b(\Omega)$, then $\rho(X_n) \searrow \rho(X)$.

If one of the two conditions above is satisfied, ρ has the robust representation

$$\rho(X) = \sup_{Q\in\mathcal{M}} \left(\mathbb{E}^Q[-X] - \alpha(Q)\right)$$

for a given penalty functional α.
If ρ is coherent and one of the two conditions above is satisfied, ρ has the robust representation

$$\rho(X) = \sup_{Q\in\mathcal{Q}} \mathbb{E}^Q[-X],$$

for a given subset $\mathcal{Q} \subseteq \mathcal{M}$.

Motivation

- The following presentation is based on the paper *Robustness in the Optimization of Risk Measures*, Embrechts, Schied and Wang, 2019.
- The main goal is to develop a methodology for determining if an optimization problem related to a risk measure is robust with respect to model uncertainty.
- Consider an n-dimensional random vector, which includes all random sources in an economic model, such as potential losses, traded securities, hedging instruments, insurance contracts, macro economic factors, or pricing densities.
- Let X be an n-dimensional random variable representing the best-of-knowledge model for an agent (e.g., computed by statistical inference) of the random vector above.
- The agent has to minimize the risk of his/her position, according to the risk measure ρ taken into consideration and to the best-of-knowledge model X along with its distribution F_X.
- Question: if X is not the true model, and the true model is another n-dimensional random variable Z, what's the residual risk that the agent is not minimizing?
- Intuitively, a risk measure ρ is robust if the solution to the risk minimization problem based on X is close to the solution to the risk minimization problem based on Z if Z is close to X (under a given pseudo-metric).

The setting

- Introduce an atomless probability space $(\Omega, \mathcal{F}, P)$. Let $L_n^0 = L_n^0(\Omega, \mathcal{F}, P)$ be the space of all P-a.s. finite, n-dimensional random variables.
- Let $\mathcal{G}_n$ be the set of measurable functions mapping $\mathbb{R}^n$ to $\mathbb{R}$.
- A random variable $g(X)$ where $g \in \mathcal{G}_n$ represents the future value of a position of an agent, according to the best-of-knowledge model X.
- The agent has to choose among admissible positions $g(X)$ for some functions g in an admissible set $\mathcal{G} \subset \mathcal{G}_n$ to minimize the risk of the position, i.e.,

  to minimize: $\rho(g(X))$ subject to $g \in \mathcal{G}$,

  with ρ risk monetary risk measure mapping a set containing $\{g(X) : g \in \mathcal{G}\}$ to $\mathbb{R} \cup \{+\infty\}$.
- Let $\mathcal{G}_X(\rho)$ be the set of optimizing functions for the model X, that is,

$$\mathcal{G}_X(\rho) = \{g \in \mathcal{G} : \rho(g(X)) = \rho(X; \mathcal{G})\},$$

  with

$$\rho(X; \mathcal{G}) = \inf\{\rho(g(X)) : g \in \mathcal{G}\}.$$

Model uncertainty

- Let $\mathcal{Z} \subset L^0_n$ be a set of possible economic vectors including X: it is interpreted as the set of alternative models.
- Introduce the pseudo-metric π^W_n in $\mathcal{Z}$ defined as

$$\pi^W_n(X, Y) = \pi_P(F_X, F_Y), \quad X, Y \in \mathcal{Z},$$

  where π_P is the Prokhorov metric over the set of probability distribution measures $\pi_P(\mu, \nu) = \inf\{\epsilon > 0 : \mu(A) \leq \nu(A_\epsilon) + \epsilon$ and $\nu(A) \leq \mu(A_\epsilon) + \epsilon$ for all $A \in \mathcal{B}(\mathbb{R}^n)\}$ where $A_\epsilon = \{x \in \mathbb{R}^n : \exists y_x \in A$ with $||x - y_x|| < \epsilon\}$ and $|| \cdot ||$ is the Euclidean norm.
- Call $Z \in \mathcal{Z}$ the real economic vector. Denote g_X a generic element $g_X \in \mathcal{G}_X(\rho)$ and g_Z a generic element $g_Z \in \mathcal{G}_Z(\rho)$.
- The real but unknown position $g_X(Z)$ may be different from the perceived optimal position $g_X(X)$.
- If Z and X are close to each other according to the pseudo-metric above, we would like $\rho(g_X(Z))$ to be close to $\rho(g_X(X))$.
- In other words, we want some continuity of the map $Y \to \rho(g_X(Y))$ at $Y = X$.

Definition

We call $(\mathcal{G}, \mathcal{Z}, \pi_n^W)$ an uncertainty triplet.

Definition

For a given uncertainty triplet $(\mathcal{G}, \mathcal{Z}, \pi_n^W)$ we say that a monetary risk measure ρ is *compatible* if ρ maps $\mathcal{G}(\mathcal{Z}) := \{g(Z) : g \in \mathcal{G}, Z \in \mathcal{Z}\}$ to $\mathbb{R} \cup \{+\infty\}$ and is distribution invariant, i.e., $\rho(X) = \rho(Y)$ if $F_X = F_Y$.

Definition

Let $(\mathcal{G}, \mathcal{Z}, \pi_n^W)$ be an uncertainty triplet. A compatible risk measure ρ is robust at $X \in Z$ relative to $(\mathcal{G}, \mathcal{Z}, \pi_n^W)$ if there exists $g_X \in \mathcal{G}_X(\rho)$ such that the function $Y \to \rho(g_X(Y))$ is π_n^W-continuous at $Y = X$.

Other possibly analysis

- The following quantities have different physical meanings:
  - $\rho(g_X(X))$: the perceived risk value optimized for X;
  - $\rho(g_Z(Z))$: the idealistic risk value optimized for Z if Z was known;
  - $\rho(g_X(Z))$: the actual risk value of the model Z, with optimization made for X.
- Correspondingly, the following quantities are given:
  - solvency gap: $\rho(g_X(Z)) - \rho(g_X(X))$;
  - optimality gap: $\rho(g_X(Z)) - \rho(g_Z(Z))$;
  - optimality shift: $\rho(g_Z(Z)) - \rho(g_X(X))$.
- Since $\rho(g_Z(Z))$ is not available, the focus is on the first quantity.

Two simple results

Remark

If $\mathcal{G}_X(\rho) = \emptyset$, ρ is not robust at X by definition.

Proposition

If $\mathcal{G}_X(\rho)$ contains a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and ρ is π_n^W-continuous, then ρ is robust at X relative to $(\mathcal{G}, \mathcal{Z}, \pi)$.

Three optimization problems for three admissible sets

- Let $n = 1$ and $X \leq 0$ be the perceived model of a random risk factor, representing a loss.
- Suppose that a position on the random risk factor can be traded in the financial market with pricing density function γ, i.e., by holding a risky position $g(X)$ one receives the monetary amount $\mathbb{E}[\gamma(-g(X))]$.
- The risk minimization problem is taken over the functions $g \in \mathcal{G}_1$ satisfying the budget constraint $\mathbb{E}[\gamma(-g(X))] \geq x_0$, for a given $x_0 > 0$.
- Consider the following three classic setups of the risk minimization problem to minimize $\rho(g(X))$ subject to $g \in \mathcal{G} \subset \mathcal{G}_1$ for the following choices for $\mathcal{G}$:
  - case with no more restrictions:

$$\mathcal{G} = \mathcal{G}_{nc} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma(-g(X))] \geq x_0\};$$

  - no short-selling or over-hedging constraint:

$$\mathcal{G} = \mathcal{G}_{ns} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma(-g(X))] \geq x_0,\ X \leq g(X) \leq 0\};$$

  - bounded constraint: for some $m > 0$,

$$\mathcal{G} = \mathcal{G}_{bc} = \{g \in \mathcal{G}_1 : \mathbb{E}[\gamma(-g(X))] \geq x_0,\ g(X) \geq -m\}.$$

Main results about Value at Risk and Average Value at Risk

Here we assume that:

- $X \in \mathcal{Z}$, $X \leq 0$ and the distribution of X has a positive density on its support.
- The pricing density $\gamma : \mathbb{R} \to \mathbb{R}^+$ is continuous and strictly positive, $\mathbb{E}[\gamma] = 1$, and $\mathbb{E}[\gamma(-X)] < \infty$.

Theorem

If γ is also nondecreasing, the Value at Risk measure is not robust at X for any of the three problems stated above.

Theorem

Suppose that either γ is a constant, or γ is a continuous function and $\gamma(-X)$ is continuously distributed. Thus, the Average Value at Risk

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_\beta(X)d\beta$$

is robust at X for any of the three problems stated above.