Economics of Risk

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Outline

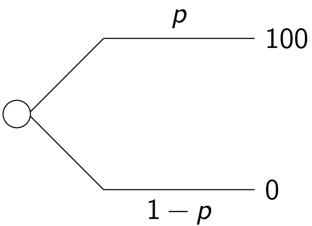
Let's go back to 1600s...

- Chevalier de Mère was a nobleman who gambled frequently
- He bet on a roll of a die that at least one 6 would appear during a total of four rolls
- From past experience, he knew that he was more successful than not in this bet
- He bet he would get a double 6 on 24 rolls of two dice
- Soon, he realized that this bet was not as profitable
- He asked his friend Blaise Pascal why
- Pascal developed a correspondence with Pierre de Fermat, and they are both credited with the founding of **probability theory**

The key idea: a bet should be evaluated by its **expected value**.

Calculating the Expected value

- Consider a bet that pays \$100 if some **event** happens, and nothing otherwise. It is represented below:



- What is its Expected Value?

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 = 100p.$$

Calculating the Expected value

- Suppose that the event is the first bet considered by de Mere, that is, to obtain a 6 if the dice is played four times. The probability of not getting 6 in four rolls is:

$$\left(\frac{5}{6}\right)^4 \approx 0.48 < \frac{1}{2}.$$

- Thus, the probability of getting at least a 6 is:

$$p = 1 - \left(\frac{5}{6}\right)^4 \approx 0.52 > \frac{1}{2}.$$

- The Expected Value of playing the bet of \$100 in this event is therefore

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 \approx 52 > 50.$$

Calculating the Expected value

- Now, assume the event is the second bet considered by de Mere, that is, to obtain a double 6 if the dice is played 24 times. The probability of not getting a double six in 24 rolls is:

$$\left(\frac{35}{36}\right)^{24} \approx 0.509 > \frac{1}{2}.$$

- Thus, the probability of winning the bet is:

$$p = 1 - \left(\frac{5}{6}\right)^{4} \approx 0.491 < \frac{1}{2}.$$

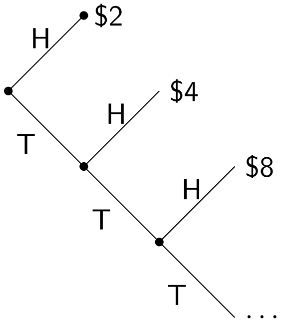
- The Expected Value of playing the bet of \$100 in this event is therefore

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 \approx 49.1 < 50.$$

So, is expectation is the answer?

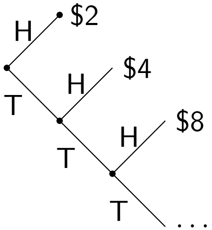
In 1713, Nicolas Bernoulli proposed the following bet:

- You toss a coin. In case of H, you receive \$2. In case of T, you toss it again.
- If H, you receive \$4. If T, you toss it again.
- In each play, the value is doubled.



A paradox!

- How much is this bet worth?
- How much would you pay to play it?



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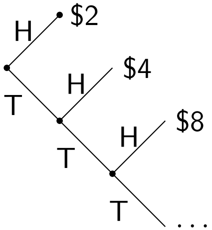


$$
\begin{aligned}
E[X] &= \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \frac{1}{16}16 + \ldots \\
&= 1 + 1 + 1 + 1 + \ldots \\
&= \infty!
\end{aligned}
$$

The birth of expected utility

- In 1738, Daniel Bernoulli (cousin of Nicolas Bernoulli), proposed a solution to the paradox:

  > *The determination of the value of an item must not be based on the price, but rather on the utility it yields... There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.*

- Daniel Bernoulli lived in St. Petersburg when he published his solution—and this is how the paradox acquired its name

How to solve

- In fact, Bernoulli proposed $u(x) = \ln(x)$.
- However, we have been using different functions. he solution Given a utility function, say $u(x) = x^{\alpha}$, for $\alpha \in (0, 1)$,
- In fact, if a person has a (reasonable) certainty equivalent C_X for X, we can find α that makes $E[(X)^{\alpha}] = C_X$.
- At least for this gamble, we could explain this individual's preference with this utility function

Outline

Mean and Variance

In the analysis of the problem of portfolio selection, Markowitz (1952) was the first to consider **mean** and **variance** to select portfolios with different assets.

- **Mean**: it is just the Expected Value (expectation) that we have been considering:

$$\text{If } X \text{ is discrete: } \mathrm{E}[X] = \sum_{i=1}^{N} p_i x_i;$$

$$\text{If } X \text{ has a p.d.f. } f_X : \mathrm{E}[X] = \int \alpha f_X(\alpha) d\alpha.$$

- **Variance** is defined by

$$\mathrm{Var}[X] \equiv \mathrm{E}\left[(X - \mathrm{E}[X])^2 \right] \geqslant 0.$$

Mean and Variance

- The Expectation has nice linearity properties, that is,

$$\mathrm{E}[\alpha X + \beta Y] = \alpha \mathrm{E}[X] + \beta \mathrm{E}[Y],$$

  for any random variables X and Y and real numbers $\alpha, \beta \in \mathbb{R}$.
- Using this property, we can simplify the expression of the variance

$$\begin{aligned}
\mathrm{Var}[X] &\equiv \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\
&= \mathrm{E}\left[X^2 - 2X\mathrm{E}[X] + (\mathrm{E}[X])^2\right] \\
&= \mathrm{E}\left[X^2\right] - 2\mathrm{E}[X\mathrm{E}[X]] + \mathrm{E}\left[(\mathrm{E}[X])^2\right] \\
&= \mathrm{E}\left[X^2\right] - 2\mathrm{E}[X]\mathrm{E}[X] + (\mathrm{E}[X])^2 \\
&= \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2.
\end{aligned}$$

Mean and Variance

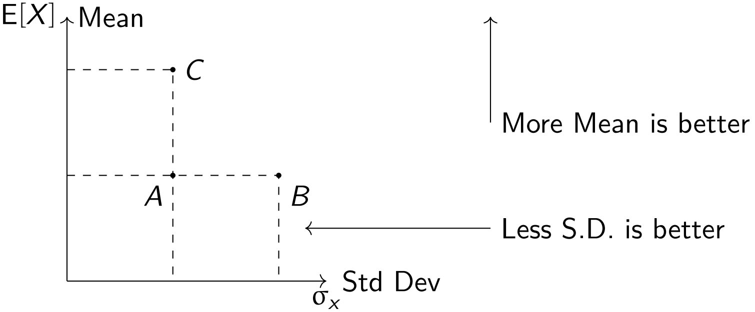
- The Variance indicates how risky an asset is. Indeed, if $X = \mathrm{E}[X]$ is risk free, we have

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2 = X^2 - X^2 = 0.$$

  (Recall that the variance is always non-negative. Thus, risk free assets have minimal variance.)
- Therefore, it is natural to think that an investor likes the mean and dislikes the variance
- With this idea, Markowitz represented assets in a graph mean vs. variance and reasoned how to compare assets with different pair (mean, variance)

Obs.: Sometimes it is convenient to use the standard deviation $\sigma_X \equiv \sqrt{\mathrm{Var}[X]}$ as a measure of the risk of the asset X, instead of just its variance $\mathrm{Var}[X]$.

Preference between Assets



- A is preferable to B because it has lower standard deviation (it is less risky) and has the same mean;
- C is preferable to A because it has higher mean and the same standard deviation;
- B is preferable to C are incomparable using only Markowitz's criteria.

Mean and Variance

- Notice that Markowitz's criteria is a methodology to compare prospects or assets;
- But we have been using Expected Utility just for that!
- Can we relate these two approaches?

Mean and Variance

- Notice that Markowitz's criteria is a methodology to compare prospects or assets;
- But we have been using Expected Utility just for that!
- Can we relate these two approaches?
- Sure! Here, I will do this exercise using only the quadratic utility function $U(x) = ax - bx^2$, for $a, b > 0$ and X with support contained in $\left[0, \frac{a}{2b}\right]$.
- Let $\mu = \mathrm{E}[X]$ and $\sigma^2 = \mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$. We have:

$$\begin{aligned} \mathrm{E}[U(X)] &= \mathrm{E}[aX - bX^2] = a\mathrm{E}[X] - b\mathrm{E}\left[X^2\right] \\ &= a\mu - b\left(\sigma^2 + \mu^2\right). \end{aligned}$$

- We can obtain the set of points with the same expected utility (indifference curve).

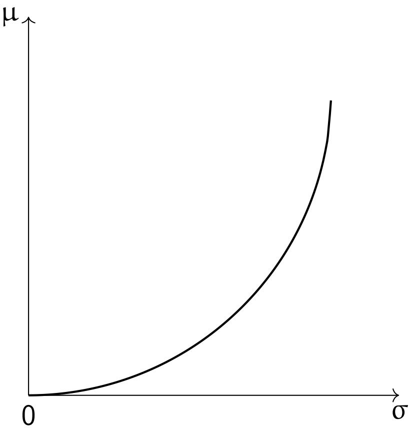
Mean and Variance

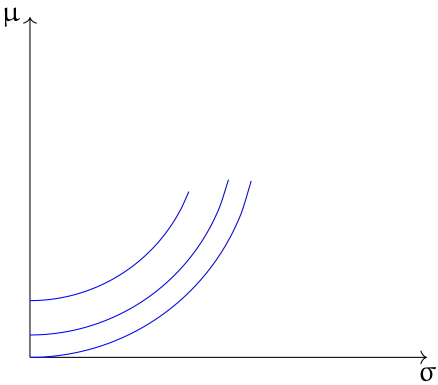
- We can obtain the set of points with the same expected utility (indifference curve):

$$\begin{aligned} \mathrm{E}[U(X)] = k &\iff a\mu - b\left(\sigma^2 + \mu^2\right) = k \\ &\iff \mu^2 - \frac{a}{b}\mu + \left(\sigma^2 + \frac{k}{b}\right) = 0 \\ &\iff \mu = \frac{a}{2b} \pm \sqrt{\left(\frac{a}{2b}\right)^2 - \left(\sigma^2 + \frac{k}{b}\right)} \end{aligned}$$

- Since the support of X is contained in $\left[0, \frac{a}{2b}\right]$, it does not make sense the "plus" signal in $\pm$ above, because this would imply $\mu > \frac{a}{2b}$.
- We can plot the above curve (with the minus sign) in the Mean-Variance axes

Mean and Variance

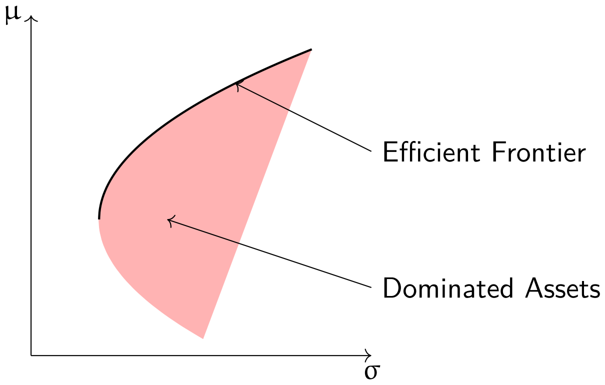




Outline

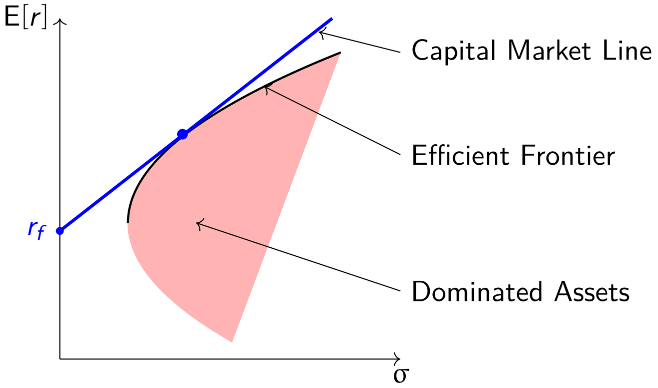
Efficient Frontier

- In any given market, there are a set of assets available, with different means and variances/standard deviations.
- Some of those assets will be dominated by preferable assets; The set of undominated assets form the "Efficient Frontier"



Capital Market Line (CML)

- If there is a risk free asset, we can connect this asset with a point in the Efficient Frontier.
- The line thus formed is called the Capital Market Line.



Capital Asset Pricing Model (CAPM)

Assumptions:

- Markets are frictionless
- Investors care only about their expected mean and variance of their returns over a given period
- Investors have homogeneous beliefs

CAPM Formula: Let M denote the market portfolio. For any asset i,

$$E[r_i] - r_f = \beta_i (E[r_M] - r_f),$$

where

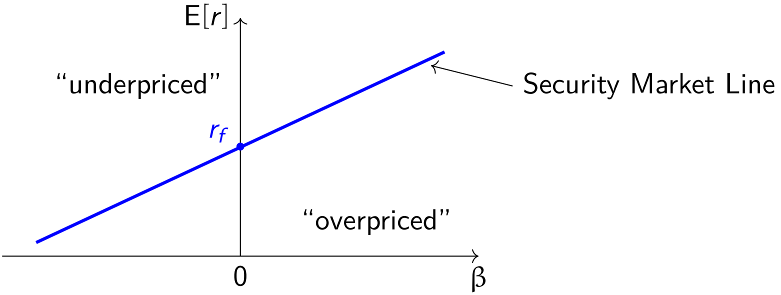
$$\beta_i = \frac{cov(r_i, r_M)}{Var[r_M]} = \frac{E[r_i r_M] - E[r_i]E[r_M]}{Var[r_M]}$$

CAPM examples

- The CAPM formula leads us to the Security Market Line (SML)

$$\mathrm{E}[r_i] - r_f = \beta_i(\mathrm{E}[r_M] - r_f),$$

- In a "CAPM world," the SML



Outline

Structure of assets

- There are n states of the Nature, $S = \{s_1, \ldots, s_n\}$.
- There are m assets.
- a_{ij} is the payment of asset $i \in \{1, \ldots, m\}$ if the state of the nature is $j \in S$.
- $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is the matrix of payoffs.
- $y = (y_1, \ldots, y_m)$ is the portfolio of assets; y_i is the quantity of asset i that is acquired (bought).
- $p = (p_1, \ldots, p_m) \in \mathbb{R}^m \setminus \{0\}$ is the vector of prices of assets; p_i is asset i's price.

Definition (Arbitrage)

We say that the above structure does not allow arbitrage if there is no $y \in \mathbb{R}^m$ such that $y \cdot A \geqslant 0$ e $p \cdot y < 0$.

Farkas' Lemma

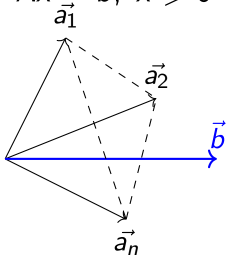
Lemma (Farkas' Lemma)

For any matrix $A \in \mathbb{R}^{m\times n}$, and vector $b \in \mathbb{R}^m$, one and only one of the following alternatives hold:

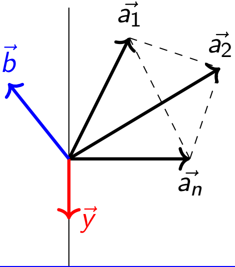
(1) $\exists\, x \in \mathbb{R}^n$, *tal que* $Ax = b, x \geqslant 0$.

(2) $\exists\, y \in \mathbb{R}^m$, *tal que* $y \cdot A \geqslant 0$ *e* $y \cdot b < 0$.

$Ax = b,\ x \geqslant 0$



$b.y < 0,\ y.A \geqslant 0$



Existence of a risk neutral probability

Definition (Arbitrage)

We say that the above structure does not allow arbitrage if there is no $y \in \mathbb{R}^m$ such that $y \cdot A \geqslant 0$ e $p \cdot y < 0$.

Definition (Risk Neutral Probability)

We say that a probability $\pi : 2^S \to [0, 1]$ is a risk neutral probability if there exists a number $\lambda \in \mathbb{R}_{++}$ such that for each asset $i \in \{1, ..., m\}$,

$$p_i = \lambda \mathrm{E}_\pi [a_i] = \lambda \sum_{j=1}^{n} \pi(\{s_j\}) a_{ij}.$$

Existence of a risk neutral probability

Theorem

There is no arbitrage if and only if there exist a risk neutral probability.

Proof of Necessity.

If there is no arbitrage, that is, there is no y satisfying $y \cdot A \geqslant 0$ and $p \cdot y < 0$, then by Farkas' Lemma $\exists\, \hat{\pi} \geqslant 0$ such that $A \cdot \hat{\pi} = p$. Since $p = (p_1, ..., p_m) \in \mathbb{R}^m \setminus \{0\}$, $\hat{\pi} \neq 0$ and $\lambda \equiv \sum_{i=1}^{n} \hat{\pi}_i > 0$. Thus, $\pi \equiv \frac{\hat{\pi}}{\lambda}$ is a probability. Therefore, for each $i = 1, ..., m$,

$$p = A \cdot (\lambda\pi) \implies p_i = \lambda \mathrm{E}_\pi[a_i] = \lambda \sum_{j=1}^{n} \pi(\{s_j\}) a_{ij},$$

as we wanted to show. □

Existence of a risk neutral probability

Proof of Sufficiency.

Let π be a risk neutral probability, that is, $p = \lambda A\pi$, for some $\lambda > 0$. Assume that there is arbitrage, that is, there exists y satisfying $y \cdot A \geqslant 0$ and $y \cdot p < 0$. However,

$$y \cdot p = y \cdot (\lambda A\pi) = \lambda y \cdot A\pi,$$

which must be non-negative because $\lambda > 0$, $y \cdot A \geqslant 0$, and $\pi(\{s_j\}) \geqslant 0$ for all j. But this contradicts $y \cdot p < 0$.

□

Outline

Conclusion

In this course, we have covered the following topics:

1. Expected values (means), variance and standard deviation
2. Expected Utility
3. Risk aversion, risk neutrality, coefficient of absolute risk aversion
4. CAPM
5. Arbitrage Pricing

Bibliography

Markowitz, H. M. (1952): "Portfolio selection," *Journal of Finance*, 7, 77–91.